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3 (Sem-3 /CBCS) MAT HC 1

2022

**MATHEMATICS**

(Honours)

Paper : MAT-HC-3016

**( Theory of Real Functions )**

Full Marks : 80

Time : Three hours

**The figures in the margin indicate full marks for the questions.**

✓ 1. Answer **any ten** parts :  $1 \times 10 = 10$

(a) Is every point in  $I$  a limit point of  $I \cap \mathbb{Q}$  ?

(b) Find  $\lim_{x \rightarrow 1} \frac{x^2 - x + 1}{x + 1}$ .

(c) Let  $f(x) = \text{sgn}(x)$ . Write the limits

$\lim_{x \rightarrow 0^+} f(x)$  and  $\lim_{x \rightarrow 0^-} f(x)$ .

Contd.



(g) Let  $p: \mathbb{R} \rightarrow \mathbb{R}$  be the polynomial function

$$p(x) := a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

if  $a_n > 0$ , then  $\lim_{x \rightarrow \infty} p(x) = ?$

(e) Let  $f$  be defined on  $(0, \infty)$  to  $\mathbb{R}$ . Then the statement

" $\lim_{x \rightarrow \infty} f(x) = L$  if and only if

$\lim_{x \rightarrow 0^+} f\left(\frac{1}{x}\right) = L$ " is true **or** false.

(f) Let  $A \subseteq \mathbb{R}$  and let  $f_1, f_2, \dots, f_n$  be function on  $A$  to  $\mathbb{R}$ , and let  $c$  be a cluster point of  $A$ . If  $\lim_{x \rightarrow c} f_k(x) = L_k$ ,

$k = 1, 2, \dots, n$ ,

then  $\lim_{x \rightarrow c} (f_1 \cdot f_2 \cdot \dots \cdot f_n) = ?$

(g) Is the function  $f(x) = \frac{1}{x}$  continuous on  $A = \{x \in \mathbb{R} : x > 0\}$ ?

(h) Write the points of continuity of the function  $f(x) = |x|$ .

(i) "A rational function is continuous at every real number for which it is defined." Is it true **or** false?

(j) "Let  $f, g$  be defined on  $\mathbb{R}$  and let  $c \in \mathbb{R}$ .

If  $\lim_{x \rightarrow c} f(x) = b$  and  $g$  is continuous

at  $b$ , then  $\lim_{x \rightarrow c} (g \cdot f)(x) = g(b)$ ." Write

whether this statement is correct or not.

(k) The functions  $f(x) = x$  and  $g(x) = \sin x$  are uniformly continuous on  $\mathbb{R}$ . Is  $fg$  uniformly continuous on  $\mathbb{R}$ ? If not, give the reason.

(l) A continuous periodic function on  $\mathbb{R}$  is bounded and \_\_\_\_\_ on  $\mathbb{R}$ .  
(Fill in the blank)

(m) "The derivative of an odd function is an even function." Write true **or** false.

(n) Write the derivative of the function  $f(x) = |x|$  for  $x \neq 0$ .



✓ (g) If  $f$  is differentiable on  $[a, b]$  and  $g$  is a function defined on  $[a, b]$  such that  $g(x) = kx - f(x)$  for  $x \in [a, b]$ . If  $f'(a) < k < f'(b)$ , then find  $g'(c)$  :- <

(p) "Suppose  $f : [0, 2] \rightarrow \mathbb{R}$  is continuous on  $[0, 2]$  and differentiable on  $(0, 2)$ , with  $f(0) = 0$ ,  $f(2) = 1$ . If there exists  $c \in (0, 2)$ , then  $f'(c) = \frac{1}{3}$ ." Is it true or false?

✓ (q) Find  $\lim_{x \rightarrow 0} \frac{x^2 + x}{\sin 2x}$ .

✓ (r) "The function  $f(x) = 8x^3 - 8x^2 + 1$  has two roots in  $[0, 1]$ ." Write true or false.

2. Answer **any five** parts :  $2 \times 5 = 10$

✓ (s) Use the definition of limit to show that  $\lim_{x \rightarrow 2} (x^2 + 4x) = 12$ .

(b) Find  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x^2}\right)$ , ( $x \neq 0$ ).

✓ (t) Give an example of a function that has a right-hand limit but not a left-hand limit at a point.

(d) Define  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) = \begin{cases} 2x & \text{for } x \in \mathbb{Q} \\ x+3, & \text{for } x \in \mathbb{Q}^c \end{cases}$$

Find all points at which  $g$  is continuous.

(e) Show that the 'sine' function is continuous on  $\mathbb{R}$ .

(f) Show that the function  $f(x) = \frac{1}{x}$  is uniformly continuous on  $[a, \infty]$ , where  $a > 0$ .

(g) Using the mean value theorem, show that

$$\frac{x-1}{x} < \ln(x) < x-1 \text{ for } x > 1.$$

(h) Show that  $f(x) = x^{1/3}$ ,  $x \in \mathbb{R}$ , is not differentiable at  $x = 0$ .



(i) Let  $f(x) = \frac{\ln(\sin x)}{\ln(x)}$

Find  $\lim_{x \rightarrow 0^+} f(x)$ .

✓ State Darboux's theorem.

3. Answer **any four** parts : 5×4=20

✓(a) Prove that a number  $c \in \mathbb{R}$  is a cluster point of a subset  $A$  of  $\mathbb{R}$  if and only if there exists a sequence  $(x_n)$  in  $A$  such that  $\lim_{n \rightarrow \infty} x_n = c$  and  $x_n \neq c$  for all  $n \in \mathbb{N}$ .

✓(b) State and prove squeeze theorem.

✓(c) Let  $A \subseteq \mathbb{R}$ , let  $f$  and  $g$  be functions on  $A$  to  $\mathbb{R}$ , and let  $f$  and  $g$  be continuous at a point  $c$  in  $A$ . Prove that  $f+g$  and  $fg$  are continuous at  $c$ .

(d) Give an example of functions  $f$  and  $g$  that are both discontinuous at a point  $c$  in  $\mathbb{R}$  such that  $f+g$  and  $fg$  are continuous at  $c$ .

(e) If  $f : A \rightarrow \mathbb{R}$  is a Lipschitz function, then prove that  $f$  is uniformly continuous on  $A$ .

(f) Determine where the function

$$f(x) = |x| + |x-1|$$

from  $\mathbb{R}$  to  $\mathbb{R}$  is differentiable and find the derivative.

(g) Find  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ .

(h) Determine whether or not  $x=0$  is a point of relative extremum of the function  $f(x) = x^3 + 2$ .

✓4. Answer **any four** parts : 10×4=40

✓(a) Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a cluster point of  $A$ . Prove that the following are equivalent :

(i)  $\lim_{x \rightarrow c} f(x) = L$



- (ii) Given any  $\varepsilon$ -neighbourhood  $V_\varepsilon(L)$  of  $L$ , there exists a  $\delta$ -neighbourhood  $V_\delta(c)$  of  $c$  such that if  $x \neq c$  is any point  $V_\delta(c) \cap A$ , then  $f(x)$  belongs to  $V_\varepsilon(L)$ .

10/1 (i) Find  $\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2}$ , where  $x > 0$ . 4

5 (ii) Prove that  $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$  does not exist but  $\lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) = 0$ . 6

(c) (i) Let  $f(x) = e^{\frac{1}{x}}$  for  $x \neq 0$ . Show that  $\lim_{x \rightarrow 0^+} f(x)$  does not exist in  $\mathbb{R}$  but  $\lim_{x \rightarrow 0^-} f(x) = 0$ . 5

(ii) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be such that  $f(x+y) = f(x) + f(y)$  for all  $x, y$  in  $\mathbb{R}$ . Suppose that  $\lim_{x \rightarrow 0} f(x) = L$  exists. Show that  $L = 0$  and then prove that  $f$  has a limit at every point  $c$  in  $\mathbb{R}$ . 5

(d) (i) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that  $f$  is not continuous at any point of  $\mathbb{R}$ . 5

(ii) Prove that every polynomial function is continuous on  $\mathbb{R}$ . 5

(e) Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$ , and let  $|f|$  be defined by  $|f|(x) = |f(x)|$  for  $x \in A$ . Also let  $f(x) \geq 0$  for all  $x \in A$  and let  $\sqrt{f}$  be defined by  $(\sqrt{f})(x) = \sqrt{f(x)}$  for  $x \in A$ . Prove that if  $f$  is continuous at a point  $c$  in  $A$ , then  $|f|$  and  $\sqrt{f}$  are continuous at  $c$ . 5+5=10

10/1 (ii) State and prove Bolzano's intermediate value theorem. 1+4=5

3 (ii) Let  $A$  be a closed bounded interval and let  $f: A \rightarrow \mathbb{R}$  is continuous on  $A$ . Prove that  $f$  is uniformly continuous on  $A$ . 5



(g) Let  $A \subseteq \mathbb{R}$  be an interval, let  $c \in A$ , and

let  $f : A \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  be functions differentiable at  $c$ . Prove that

(i) the function  $f + g$  is differentiable at  $c$  and

$$(f + g)'(c) = f'(c) + g'(c) \quad 5$$

(ii) if  $g(c) \neq 0$ , then the function  $\frac{f}{g}$  is differentiable at  $c$  and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad 5$$

(h) State and prove Rolle's theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(i) Use Taylor's theorem with  $n=2$  to approximate  $\sqrt[3]{1+x}$ ,  $x > -1$ . 5

(ii) If  $f(x) = e^x$ , show that the remainder term in Taylor's theorem converges to zero as  $n \rightarrow \infty$  for each fixed  $x_0$  and  $x$ . 5

(i) Find the limits :

5+5=10

(i)  $\lim_{x \rightarrow 0^+} x^{\sin x}$

(ii)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\sec x}$

